

The Berry Phase and Monopoles in Gluodynamics*

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We introduce a gauge invariant definition of a monopole on the lattice. The construction is based on the observation that for each Wilson loop there exists an extra $U(1)$ group which leaves the loop invariant. Since the lattice formulation utilizes the language of Wilson loops, the definition of the monopole charge in terms of this plaquette dependent $U(1)$ is gauge invariant. The explicit construction of gauge invariant monopoles is presented both in continuum and on the lattice.

Introduction of magnetic monopoles in non-Abelian pure gauge models has a long history (see [1] and references therein). The basic problem is that the monopoles are intrinsically $U(1)$ objects and, in absence of the physical Higgs field, the choice of the $U(1)$ turns out to be a matter of gauge fixing. The standard procedure² to define monopole charge is to partially (up to a remaining $U(1)$) fix the gauge and then construct Abelian monopoles in this particular gauge. Due to the ambiguity of gauge fixing prescription the construction is not unique and it is a separate question which monopole definition is physically relevant.

1. The crucial observation [2] which is used in considerations below is that in lattice formulation there is in fact a natural $U(1)$ subgroup locally embedded into $SU(2)$. Indeed, the action of $SU(2)$ lattice gluodynamics is constructed in terms of elementary Wilson loops $U_p = e^{i\vec{F}_p\vec{\sigma}/2} = e^{i[F_p|\vec{n}_p\vec{\sigma}/2]}$, $\vec{n}_p^2 = 1$ and therefore possesses an additional symmetry $\vec{F}_p \rightarrow \vec{F}_p + 4\pi\vec{n}_p$ which is evidently lost in the naive continuum limit. In other words Wilson loop itself defines a natural $U(1)$ associated with it as the group of rotations around \vec{n}_p . Since these Abelian rotations are defined in terms of a Wilson loop, the definition of the $U(1)$ is gauge invariant by construction. The application of this procedure to each plaquette gives, basically, a gauge invariant fixation of $U(1)$ which

is plaquette dependent.

To make the next step and define monopoles in terms of this local, gauge invariant $U(1)$ we utilize the interpretation of Wilson loop as an evolution operator of quantum mechanical system [3]. As follows from the standard representation $W(C) = W(T) = \text{P exp}\{i\oint_0^T A dt\}$ the Hamiltonian governing evolution is $H = -A$ and quantum mechanical state space coincides with irreducible representation space of $SU(2)$. Here C is an arbitrary closed contour parameterized by $t \in [0; T]$. Furthermore, we introduce the conventional [4] coherent state basis $\{|\vec{n}\rangle\}$ parameterized by a set of unit three-dimensional vectors \vec{n} , $\vec{n}^2 = 1$. The action of an arbitrary element $g \in SU(2)$ in this basis is³ $g|\vec{n}\rangle = e^{i\varphi}|\vec{n}'\rangle$. A special case of cyclic evolution

$$W(T)|\vec{n}(0)\rangle = e^{i\varphi(T)}|\vec{n}(0)\rangle \quad (1)$$

is of particular importance. In Eq. (1) the evolving state $|\vec{n}(t)\rangle$ is determined by

$$\text{P exp}\{i\int_0^t A d\tau\}|\vec{n}(0)\rangle = e^{i\varphi(t)}|\vec{n}(t)\rangle. \quad (2)$$

Clearly the cyclic state $|\vec{n}\rangle$ always exists and moreover $\frac{1}{2}\text{Tr}W(T) = \cos\varphi(T)$. From (2) it directly follows that

$$\varphi(T) = \int_0^T \left(\langle \vec{n} | A | \vec{n} \rangle + i \langle \vec{n} | \frac{d}{dt} | \vec{n} \rangle \right) dt = \quad (3)$$

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²Throughout this paper we consider the case of $SU(2)$ gauge group only.

³For simplicity we consider fundamental representation only, generalization to higher representations is straightforward.

$$= \int_C \vec{n} \vec{A} + \frac{1}{2} \int_{S_C} \vec{n} \cdot [\partial \vec{n} \times \partial \vec{n}],$$

where the state $|\vec{n}\rangle$ has been smoothly extended from the contour C into an arbitrary surface S_C bounded by C . Note that Eq. (3) cannot be used in the Wilson loop calculation since construction of the evolving state $|\vec{n}\rangle$ requires knowledge of the Wilson loop itself. Nevertheless, Eq. (3) is useful theoretically since it represents the phase angle of Wilson loop as an integral of abelian 't Hooft tensor [5]. Note also that the angle $\varphi(T)$ is well defined only modulo 2π . Indeed, one can verify explicitly that φ is changing $\varphi \rightarrow \varphi + 2\pi k$, $k \in \mathbb{Z}$ under gauge transformations. From now on we always take $\varphi(T)$ in “fundamental domain” $-\pi < \varphi \leq \pi$.

Consider now the infinitesimal version of Eq. (3) when contour C is the boundary of elementary surface element $\delta\sigma$

$$\varphi = \left(\partial \wedge (\vec{n} \vec{A}) + \vec{n} \wedge \partial \vec{n} \wedge \partial \vec{n} \right) \delta\sigma. \quad (4)$$

It is straightforward now to integrate (4) over arbitrary closed two-dimensional surface S_{phys}^2 in physical space. Usual assumption of fields continuity guarantees that \vec{n} is a smooth field on S_{phys}^2 . Therefore integral is non-zero in general

$$\int_{S_{phys}^2} \varphi = 2\pi Q, \quad Q \in \mathbb{Z} \quad (5)$$

due to the second term in Eq. (4) which is widely known as Berry phase [6]. In fact Eq. (5) calculates the gauge invariant monopole charge Q contained inside S_{phys}^2 . Gauge invariance is evident since integral is constructed in terms of infinitesimal Wilson loops only. The term “monopole charge” is also quite natural because Eq. (5) coincides with well known Abelian monopole construction when gauge fields have the same color structure. Moreover, Eq. (5) being considered in four-dimensional space-time defines a closed world-lines of topological defects which have a natural interpretation of monopole trajectories.

To conclude this section we would like to emphasize that gauge invariant monopole charge definition is only possible when gluodynamics is considered as limiting case of lattice gauge models because of periodicity of gauge fields action

on the lattice (or more details see [2,7]). One can verify that Eq. (5) considered in the context of conventional continuum gluodynamics is trivial and produces identically zero magnetic charge.

2. Being extremely simple in continuum limit, Eq. (5) is quite non-trivial to implement on the coarse lattice. The problem is that lattice discretization is not suitable to calculate most of topological invariants (e.g. instanton number). In our case the problem is even more severe since effective Higgs field in Eq. (4,5) is defined not in lattice sites as usual, but on elementary 2-cells (plaquettes). Below we illustrate the monopole charge calculation in case of single three-dimensional cube. More realistic calculations will be presented elsewhere [7].

To begin with we consider a single plaquette situated at lattice site x and directed along μ , ν space-time directions. It is straightforward to calculate plaquette matrix U_p as ordered product of links. For fundamental representation there are two eigenvectors $|\vec{n}_{\pm}\rangle$: $U_p|\vec{n}_{\pm}\rangle = e^{\pm i\varphi}|\vec{n}_{\pm}\rangle$ which are related by $\vec{n}_+ = -\vec{n}_-$. While only a single plaquette is considered there is no way to distinguish between $|\vec{n}_+\rangle$, $|\vec{n}_-\rangle$ and one can take either of them as initial state $|\vec{n}_1\rangle$ to be ascribed to point 1 on the plaquette (see Fig. 1). Starting from $|\vec{n}_1\rangle$ one builds the corresponding evolving states $|\vec{n}_i\rangle$ in all other plaquette corners: $U_{\mu}(x)|\vec{n}_1\rangle = e^{i\varphi_1}|\vec{n}_2\rangle$ etc. Thus the lattice implementation of Eq. (3,4) is obtained.

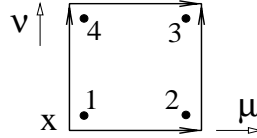


Fig. 1

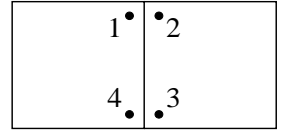


Fig. 2

Consider now the intersection of two plaquettes, Fig. 2. Evidently, on coarse lattice the states build separately on each plaquette differ drastically. In particular, the states $|\vec{n}_i\rangle$, $i = 1, 2$ and $i = 3, 4$ nearest to common link do not form continuous vector field \vec{n} which is needed to go from Eq. (4) to Eq. (5). Moreover, there is also a mentioned ambiguity in the choice of initial states $|\vec{n}_{\pm}\rangle$. Since the problem is only due to the lattice coarseness there should be no differ-

ence in continuum limit between various ways to overcome it. We propose to introduce additional two dimensional cells in the intersection of every two plaquettes which do not lie in the same plane, Fig. 3. The newly introduced links V_{12} , V_{34} may be defined in fact unambiguously. Indeed, among various $SU(2)$ matrices which satisfy $V_{12}|\vec{n}_2\rangle = e^{i\alpha}|\vec{n}_1\rangle$ there is only one which corresponds to geodesic motion $\vec{n}_2 \rightarrow \vec{n}_1$. Note that these new links are in adjoint representation since for example \vec{n}_1 and \vec{n}_2 transform in the same

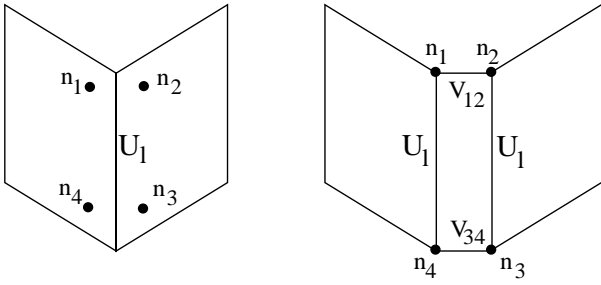
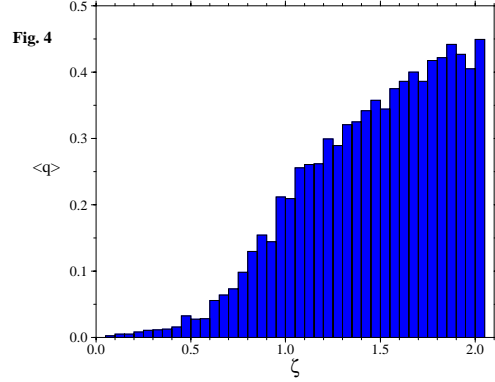


Fig. 3

way under gauge transformation: $|\vec{n}_i\rangle \rightarrow g|\vec{n}_i\rangle$, $i = 1, 2$. The ambiguity in choice of initial state on every plaquette may be resolved in similar manner. Namely, one can impose a requirement that resulting field \vec{n} should be the smoothest one.

To test the proposed algorithm we have considered the simplest case of single three-dimensional cube. The first test was to generate random pure Abelian gauge fields, calculate the monopole charge in standard way and then apply our method after random non-Abelian gauge transformation. We found that our method works perfectly on single 3-cube reproducing the known monopole charge in all cases. Furthermore, we have found a reasonable behavior when non-Abelian gauge fields $U_l = \text{const} \cdot \{1 + \zeta i \sum_{i=0}^3 \varepsilon_i \sigma^i\}$ were generated (here $\varepsilon_i \in [-1; 1]$ are random numbers). On Fig. 4 the average monopole charge as a function of ζ is shown.

3. We have proposed a gauge invariant definition of Abelian monopole charge in gluodynamics which is turn is only possible when gluodynamics is considered as limiting case of lattice gauge models. The crucial element of the presented construction is the observation that actual symmetry group of $SU(2)$ LGT is in fact $SU(2) \times U(1)$ and



therefore monopoles defined with respect to second factor are $SU(2)$ invariant. The explicit calculations are carried out both in continuum and on the lattice. Being transparent in continuum limit the actual calculations become rather intricate on the coarse lattice. We have proposed a way to overcome this difficulty which allows to investigate the monopole dynamics numerically. Unfortunately, we were not yet able to implement the method in realistic calculations since it requires changing of lattice geometry and standard Monte-Carlo algorithms. The work in these directions is currently in progress.

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